THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. JANUARY, 1898. No. 1.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from December Number.]

PROPOSITION XXXI. *Now I say there will be, of the aforesaid common perpendiculars in two distinct points, no determinate limit, such that under a smaller and smaller acute angle made at the point A, it would not always be possible to attain (in hypothesis of acute angle) to such a common perpendicular in two distinct points as is less than any assignable length R.*

PROOF. For in so far as the thing were otherwise; if from the point K (resume Fig. 30.) in BX assigned at any however great distance from the point B, a perpendicular KL is erected, to which from point A (by Euclid I. 12) the perpendicular AL is supposed let fall, KL ought to be greater than the length R.

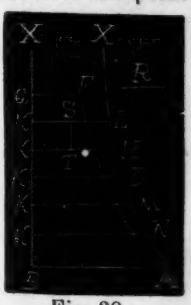

Fig. 30.

The reason is; because a higher point Q being assumed in this BX, from which is erected to BX the perpendicular QF, to which (by the same Euclid I. 12) a perpendicular AF is let fall, this again must anyhow not be less than the length R.

But KL (from Corollary to preceding Proposition) will be greater than QF. Therefore KL would be greater than the aforesaid length R. And so ever proceeding higher.

But now, if this however great KB is supposed divided (as in Proposition XXV) into portions KK, equal to the length R, and from these points K perpendiculars are erected, which meet AX in points H, D, M; the angles at these points, toward the parts of the point L, will neither be right nor obtuse; lest in some quadrilateral, as suppose $KMLK$, the four angles together should be equal to or greater than four rights, contrary to the hypothesis of acute angle, according to which we are proceeding. Therefore all such angles will be acute toward the parts of the point L; and therefore in like manner all at these points obtuse toward the parts of the point A. Wherefore (from Corollary I to Proposition III) of the aforesaid perpendiculars the least will indeed be KL more remote from the base AB, the greatest KM nearer this base.

And of the remaining the nearer will be ever greater than the more remote.

Therefore (from the preceding Proposition XXV, and its Corollary) the four angles together of the quadrilateral $KHLK$ more remote from base AB will be greater than the four angles together of all the remaining quadrilaterals nearer to this base. Wherefore (as in Proposition XXV) the hypothesis of acute angle would be destroyed.

Therefore it holds, that of the aforesaid common perpendiculars in two distinct points there will be no determinate limit, such that under a smaller and smaller acute angle made at the point A, it would not always be possible to attain (in hypothesis of acute angle) to such a common perpendicular in two distinct points as may be less than any assigned length R.

Quod erat demonstrandum.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton, New Jersey.

IV.

PROOF OF LIE'S THEOREM THAT A ONE PARAMETER GROUP CONTAINS BUT ONE INFINITESIMAL TRANSFORMATION AND ITS CONVERSE THEOREM. EXAMPLES.

13. In the preceding paragraphs it has been shown by methods of proof due to LIE that every one parameter group with inverse transformations contains an infinitesimal transformation and conversely, every infinitesimal transformation generates a one parameter group. It is the purpose of this paragraph to present the proof of the theorem that the indefinite article "a" in these theorems can be replaced by the definite modifier "one and but one." The theorem

is necessary to a rigorous grounding of the fundamental details of the theory of the group of one parameter ; the proof is less simple than the proofs of the previous theorems because it makes use of an elementary theorem of the theory of functions.

Consider again the G_1

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a). \tag{1}$$

We have found that every G_1 contains an infinitesimal transformation ; hence we may assume the existence of an infinitesimal transformation, say

$$x'=x+\xi(x, y)\delta t+\ldots\ldots, \quad y'=y+\eta(x, y)\delta t+\ldots\ldots, \tag{2}$$

belonging to the G_1, (1).

Let T_a be the transformation of the group (1) corresponding to the value a of the parameter and carrying the point (x, y) into the position (x_1, y_1). Let I be the infinitesimal transformation (2) of the group (1) and let it change the point (x_1, y_1) into the point (x_2, y_2) given by the equations

$$I \qquad x_2=x_1+\xi(x_1, y_1)\delta t+\ldots\ldots, \quad y_2=y_1+\eta(x_1, y_1)\delta t+\ldots\ldots. \tag{3}$$

The transformation equivalent to the product T_aI transforms the point (x, y) into the point (x_2, y_2) and is found by eliminating (x_1, y_1) from the equations (3) by means of the equations (1). This elimination partially effected gives

$$S\equiv T_aI\equiv T_{a+\delta a}\begin{cases} x_2=\varphi(x, y, a)+\xi(x_1, y_1)\delta t+\ldots\ldots, \\ y_2=\psi(x, y, a)+\eta(x_1, y_1)\delta t+\ldots\ldots, \end{cases} \tag{4}$$

where the x_1 and y_1 allowed to remain are to be expressed in terms of x, y, and a by means of the equations (1).

The equations (4) represent the transformation S which is equivalent to the successive application of T_a and I. Since T_a and I belong to the group (1), their product S belongs, by definition, to the group (1). Since I is an infinitesimal, the product, S, of T_a and I differs by an infinitesimal from T_a and accordingly the parameter of S has a value, $a+\delta a$, differing by an infinitesimal from the parameter, a, of T_a, where δa is an infinitesimal quantity of the same nature as δt in the equations (2). Further, we have proved that if a is the parameter of a transformation of a given group and a_1 the parameter of a second transformation of the same group, the parameter α of their product is a function of a and a_1 alone. The parameter of T_a is a, that of I is δt, and that of S, the product of T_a and I, is $a+\delta a$; hence $a+\delta a$ is a function of a and δt alone, *i. e.* δa depends on a and δt alone.

The transformation $T_{a+\delta a}$ which is equivalent to S and is a member of the group (1) has the form

$$x_2=\varphi(x, y, a+\delta a), \quad y_2=\psi(x, y, a+\delta a), \tag{5}$$

or developed in powers of δa,

$$x_2=\varphi(x, y, a)+\frac{\partial\varphi(x, y, a)}{\partial a}\delta a+\ldots\ldots,$$
$$y_2=\psi(x, y, a)+\frac{\partial\psi(x, y, a)}{\partial a}\delta a+\ldots\ldots. \tag{6}$$

Comparing these expressions (6) for x_2, y_2 with the forms given by the equations (4) we have

$$\begin{cases}\xi(x_1, y_1)\delta t+\ldots\ldots=\dfrac{\partial\varphi(x, y, a)}{\partial a}\delta a+\ldots\ldots,\\ \eta(x_1, y_1)\delta t+\ldots\ldots=\dfrac{\partial\psi(x, y, a)}{\partial a}\delta a+\ldots\ldots.\end{cases} \tag{7}$$

In these equations (7), x_1 and y_1 are definite functions of x, y, and a, and conversely, x and y are definite functions of x_1, y_1, and a, given by the equations (1) ; δa and δt are infinitesimals and, as is shown above, δa is a function of a and δt ; the equations are true for all values of x, y, and a.

If in the equations (1) x and y are given any definite numerical values, x_1 and y_1 depend only on a. Hence if x and y are given any definite numerical values in the equations (7), these equations will express relations in δt, δa and a alone. These relations must of course agree with the one that expresses δa in terms of a and δt. Now we can choose the numbers x_1 and y_1 so that the first coefficients of one of the two relations, namely the quaatities

$$\xi(x_1, y_1),\quad \frac{\partial\varphi(x, y, a)}{\partial a}, \tag{8}$$

or the quantities

$$\eta(x_1, y_1),\quad \frac{\partial\psi(x, y, a)}{\partial a}, \tag{9}$$

do not vanish.*

Then the first or second equation (7) gives the relation between δa, δt and a in the form

*It is obvious that both of the quantities $\xi(x_1, y_1)$ $\eta(x_1, y_1)$ cannot vanish identically, else we should have no transformation. We may assume then that $\xi(x_1, y_1)$ does not vanish. It is clear then that $\frac{\partial\varphi(x, y, a)}{\partial a}$ cannot, in general, vanish ; for if it should vanish identically for all values of x and y, $\varphi(x, y, a)$ would be free from a and hence x would not be transformed at all, *i. e.* $\xi(x_1, y_1)$ would be identically zero ; but the latter is contrary to hypothesis.

$$u_1\delta t+u_2\delta t^2+\ldots\ldots=v_1\delta a+v_2\delta a^2+\ldots\ldots, \qquad (10)$$

in which $u_1, u_2, \ldots\ldots, v_1, v_2, \ldots\ldots$ depend upon a. u_1, v_1 are both different from zero since they replace the quantities (8) or the quantities (9).

Now it is a theorem of the theory of functions that when two quantities δa and δt are related as in (10), δa may be developed in a power series of δt, whose first coefficient is not zero, *i. e.*

$$\delta a=w_1\delta t+w_2\delta t^2+\ldots\ldots,$$

where w_1 is not identically zero. The w_i are certain functions of a. Substituting this value of δa in the equations (7), conceiving x_1 and y_1 as variables again, we have

$$\begin{cases} \xi(x_1, y_1)\delta t+\ldots\ldots=\dfrac{\partial\varphi(x, y, a)}{\partial a}(w_1\delta t+w_2\delta t^2+\ldots\ldots)+\ldots\ldots, \\ \eta(x_1, y_1)\delta t+\ldots\ldots=\dfrac{\partial\psi(x, y, a)}{\partial a}(w_1\delta t+w_2\delta t^2+\ldots\ldots)+\ldots\ldots \end{cases} \qquad (11)$$

Dividing through by δt and passing to the limit $\delta t=0$, these become

$$\begin{cases} \xi(x_1, y_1)=\dfrac{\partial\varphi(x, y, a)}{\partial a}w_1(a), \\ \eta(x_1, y_1)=\dfrac{\partial\psi(x, y, a)}{\partial a}w_1(a), \end{cases} \qquad (12)$$

The equations (1) solved for x and y give

$$x=\lambda(x_1, y_1, a), \quad y=\mu(x_1, y_1, a). \qquad (13)$$

If these values of x and y be put in the equations (12) we have

$$\begin{cases} \xi(x_1, y_1)=X(x_1, y_1, a)w_1(a), \\ \eta(x, y_1)=Y(x_1, y_1, a)w_1(a). \end{cases} \qquad (14)$$

The equations (14) must be true for all valoes of x_1, y_1 and a. Their left members do not contain a, hence their right members do not really contain a, but only in appearance, *i. e.* the functions X and Y have the form

$$X(x_1, y_1, a)\equiv\frac{A(x_1, y_1)}{w(a)}, \quad Y(x_1, y_1, a)\equiv\frac{B(x_1, y_1)}{w(a)}. \qquad (15)$$

If we give now to the quantity a in the equations (14) a definite value*$\overline{a}$, the functions X and Y are changed into functions of x_1 and y_1 alone,

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*For example, a solution of the equation $w(a)-1=0$ would be such a value of a that would reduce the functions X and Y to functions of x_1 and y_1 alone.

$$X(x_1, y_1, \bar{a}) \equiv \bar{X}(x_1, y_1), \quad Y(x_1, y_1, \bar{a}) \equiv \bar{Y}(x_1, y_1). \tag{16}$$

$w_1(a)$ becomes $w_1(\bar{a})$, which is a constant, since $\bar{a}$ is a definite number. Then, excepting this constant factor, the defining functions $\xi(x_1, y_1)$ and $\eta(x_1, y_1)$ of the infinitesimal transformation are completely determined, that is, if we call this constant k, we have

$$\xi(x_1, y_1) = k\,\bar{X}(x_1, y_1), \quad \eta(x_1, y_1) = k\,\bar{Y}(x_1, y_1). \tag{17}$$

This result obtains for *every* infinitesimal transformation (2) of the G_1 (1).

Any two infinitesimal transformations of the group (1), say

$$x' = x + \xi\delta t + \ldots\ldots, \quad y' = y + \eta\delta t + \ldots\ldots$$

and (18)

$$x' = x + \bar{\xi}\delta t + \ldots\ldots, \quad y' = y + \bar{\eta}\delta t + \ldots\ldots$$

can differ in their terms of the first order only by a constant factor, since

$$\bar{\xi} \equiv k\xi, \quad \bar{\eta} \equiv k\eta.$$

LIE calls two such infinitesimal transformations, whose terms of the first order differ only by a constant factor, *dependent* infinitesimal transformations since they are essentially the same, for, inasmuch as δt was taken as an arbitrary infinitely small quantity, $k\delta t$ has the same meaning as δt.

In this manner LIE arrives at the theorem: *A one parameter group of the plane with inverse transformations contains one infinitesimal transformation and no more; or, more accurately expressed, all infinitesimal transformations of a one parameter group agree in their terms of the first order excepting a constant factor.**

14. The factor $w_1(a)$ may be gotten rid of in the equations (12) by introducing a new parameter in the group (1) in place of a. If a_0 is the value of the parameter a which gives the identical transformation of the group (1), the new parameter, t, is given by

$$t = \int_{a_0}^{a} \frac{da}{w_1(a)}, \tag{19}$$

and the equations of the group become

$$x_1 = \Phi(x, y, t), \quad y_1 = \Psi(x, y, t). \tag{20}$$

Since $a = a_0$ makes $t = 0$ in (19), it is clear that in the form (20) the identical transformation, $x_1 = x$, $y_1 = y$, is given by the value of the parameter t=0. Hence the equations (12) may be written

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*See LIE—*Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, bearbeitet und herausgegeben von Dr. Georg Scheffers, Leipzig, 1891, pp. 38 et seq.

$$\left\{\begin{array}{l} \xi(x_1, y_1)=\dfrac{\partial\varphi(x, y, a)}{\partial a}\dfrac{da}{dt}\equiv\dfrac{\partial\Phi(x, y, t)}{\partial t}, \\ \eta(x_1, y_1)=\dfrac{\partial f(x, y, a)}{\partial a}\dfrac{da}{dt}\equiv\dfrac{\partial\Psi(x, y, t)}{\partial t}, \end{array}\right. \tag{21}$$

or
$$\xi(x_1, y_1)=\frac{dx_1}{dt}, \quad \eta(x_1, y_1)=\frac{dy_1}{dt}, \tag{22}$$

since x_1, y_1 are the functions (1) of a or the functions (20) of t.

Hence the equations (20), which are equivalent to the original equations (1) of the group, are the integral equations of the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)}=\frac{dy_1}{\eta(x_1, y_1)}=dt, \tag{23}$$

with the initial conditions $x_1=x$, $y_1=y$, $t=0$.

Since this simultaneous system and its integral equations are completely determined when the first members

$$x'=x+\xi(x, y)\delta t, \quad y'=y+\eta(x, y)\delta t,$$

of the infinitesimal transformation of the group are given, LIE has the theorem—*a one parameter group of the plane is completely defined by its infinitesimal transformation; or otherwise expressed, every infinitesimal transformation belongs to one one parameter group of the plane and no more.*

This theorem and the preceding one are incorporated in the following second fundamental theorem of LIE's theory of the group of one parameter.

THEOREM: *Every one parameter group of the plane whose transformations are inverse in pairs contains one infinitesimal transformation and no more. Every infinitesimal transformation of the plane belongs to one one parameter group and to one only. The latter's transformations are inverse in pairs.*

The reader may find it to his advantage to work through the following examples carefully. They are in illustration of the theorems of the last two notes of this series and are drawn from LIE's lectures on differential equations.

1°. If the given infinitesimal transformation has the form

$$x_1=x+x\delta t, \quad y_1=y+y\delta t,$$

then
$$\xi(x, y)\equiv x, \quad \eta(x, y)\equiv y,$$

and the simultaneous system is

$$\frac{dx_1}{x_1}=\frac{dy_1}{y_1}=dt.$$

The integration of this simultaneous system gives

$$\log x_1 - \log x = \log y_1 - \log y = t,$$

or $$x_1 = e^t x, \quad y_1 = e^t y.$$

These equations represent a one parameter group as may be readily verified directly. Any transformation of the group changes all abscissas and ordinates in the same ratio, *i. e.* the entire plane is proportionately increased or diminished from the origin of coördinates. $t=0$ gives the infinitesimal transformation of the group. By the exponential theorem we have

$$e^{\delta t} = 1 + \frac{\delta t}{1!} + \frac{\delta t^2}{2!} + \dots\dots;$$

hence the infinitesimal transformation of the group is

$$x_1 = (1 + \frac{\delta t}{1!} + \frac{\delta t^2}{2!} + \dots\dots)x, \quad y_1 = (1 + \frac{\delta t}{1!} + \frac{\delta t^2}{2!} + \dots\dots)y,$$

which agrees, in its terms of the first order, with the original infinitesimal transformation.

2°. The equations

$$x_1 = \sqrt{x^2 + xyt}, \quad y_1 = \frac{xy}{\sqrt{x^2 + xyt}}$$

represent a one parameter group, as appears in the following manner by making use again of the fundamental theorem of note III. We have clearly

$$x_1 y_1 = xy \quad \text{and} \quad \frac{x_1}{y_1} = \frac{x^2 + xyt}{xy} \equiv \frac{x}{y} + t.$$

These equations have the form of those in the theorem named if we put

$$\Omega(x, y) \equiv xy, \quad W(x, y) \equiv x/y.$$

$t=0$ corresponds to the identical transformation of the group, hence the infinitesimal transformation has the parameter $t=\delta t$. Since

$$\sqrt{x^2 + xyt} = x\sqrt{1 + \frac{y}{x}\delta t} = x(1 + \tfrac{1}{2}\frac{y}{x}\delta t + \dots\dots),$$

the infinitesimal transformation of the group is represented by tne equations

$$x_1 = x + \tfrac{1}{2}y\delta t + \dots\dots, \quad y_1 = y - \tfrac{1}{2}\frac{y^2}{x}\delta t + \dots\dots.$$

Hence, $$\xi(x, y) \equiv \tfrac{1}{2}y, \quad \eta(x, y) \equiv -\tfrac{1}{2}\frac{y^2}{x},$$

and the finite equations of the group will appear again by integrating the simultaneous system

$$\frac{2dx_1}{y_1} = -\frac{2x_1 dy_1}{y_1^2} = dt.$$

3°. x_1 and y_1, the roots u of the quadratic equation

$$(u-x)(u-y)+t=0,$$

expressed in terms of x, y and t, define a G_1.

The equation may be written

$$u^2-(x+y)u+xy+t=0.$$

Then by an elementary theorem of the theory of algebraic equations,

$$x_1+y_1=x+y,$$

$$x_1y_1=xy+t.$$

These are two equations of the form

$$\Omega(x_1, y_1)=\Omega(x, y), \quad W(x_1, y_1)-t=W(x, y),$$

and hence represent a group.

By actually solving the equations it may readily be shown that thatt he infinitesimal transformation of the group is given by the equations

$$x_1=x+\frac{\delta t}{y-x}+\ldots\ldots, \quad y_1=y+\frac{\delta t}{x-y}+\ldots\ldots$$

4°. The equations

$$x_1=x+t, \quad y_1=\frac{xy-t}{x+t}$$

represent a G_1. In this case

$$\xi(x, y)\equiv, \quad \eta(x, y)\equiv-\frac{1+y}{x}.$$

The finite equations are integral equations of the simultaneous system

$$\frac{dx_1}{1} = -\frac{x_1 dy_1}{1+y_1} = dt.$$

The reader will have no trouble in verifying these results.

Princeton University, 6 January, 1898.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

84. Proposed by SYLVESTER ROBBINS, North Branch, New Jersey.

Show how to find sides, integral, fractional, and irrational, for twenty-four triangles, each one containing 330 square yards.

Partial solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics in Drury College, Springfield, Mo.

This is properly a problem of Diophantine Analysis, and should have been published under that department.

Let a, b, c, $a>b>c$, be three integers having no common divisor, and let them represent the sides of the triangle. Then the area of a triangle in terms of its sides is

$$\triangle=\tfrac{1}{4}\sqrt{2\Sigma a^2b^2-\Sigma a^4}.$$

Solving this equation for a, we have,

$$a=\sqrt{b^2+c^2\pm 2\sqrt{b^2c^2-4\triangle^2}}.$$

Let $\sqrt{b^2c^2-4\triangle^2}=bc-2\triangle v$. Then $bc=(\triangle/v)(v^2+1)$ where v can be any number so far as making the quantity under the radical a perfect square is concerned.

$\therefore\ a=\sqrt{b^2+c^2\pm 2\triangle/v(v^2-1)}$.

If the area of a triangle is expressed by an integer the following principles can be easily proved:

(1) The sides can not all be even.

(2) The sides can not all be odd.

(3) No two sides can be even.

COROLLARY 1. Hence, of the three sides one must be even and the other two odd.

COROLLARY 2. The perimeter of the triangle is expressed by an even number.

If we assume b and c to represent odd integers, and since $bc=(\triangle/v)(v^2+1)$, we may find values of b and c by giving v such values as will make $(\triangle/v)(v^2+1)$ an odd integer. Then by taking such factors of this integer for values of b and c as will render $\sqrt{b^2+c^2+2(\triangle/v)(v^2+1)}$, a perfect square, we get integral values required. For example, if $v=2$, $b=33$, and $c=25$, gives $a=52$. But this does not furnish a ready means of writing out integral values.

If any one will furnish us a more expeditious method we shall be pleased to publish it in the next issue of the MONTHLY.

85. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

In turning a one-horse chaise within a ring of a certain diameter, it was observed that the outer wheel made two turns, while the inner wheel made but one. The wheels were each 4 feet high; and supposing them fixed at the distance of 5 feet on the axletree, what was the circumference of the track described by the outer wheel? From *Greenleaf's National Arithmetic.*

Solution by EDWIN R. ROBBINS, Master of Mathematics, Lawrenceville School, Lawrenceville, N. J.

Let r=radius, then $2\pi r$=circumference of inner track.

$r+5$=radius, then $2\pi(r+5)$=circumference of outer track.

From problem, $2(2\pi r)=2\pi(r+5)$, whence $r=5$.

Hence, outer track$=20\pi=62.83184+$ feet.

Also solved by *J. SCHEFFER, G. B. M. ZERR, F. R. HONEY, CHARLES C. CROSS, LEE WILCOX,* and *P. S. BERG.*

86. Proposed by EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Ga.

$\frac{1}{7}=.\dot{1}42857\dot{1}$; $\frac{1}{11}=.\dot{0}\dot{9}$; $\frac{1}{13}=.\dot{0}76923\dot{0}$; $\frac{1}{17}=.\dot{0}58823594117647\dot{0}$.

Observe that if the numbers forming the first half of the repetend be added respectively to the numbers forming the second half of the repetend, the sum is in every case 9. What is the general law of which these are special cases?

I. Solution by O. W. ANTHONY, M. Sc. Instructor in Mathematics in Boys' High School, New York City.

Put $\frac{1}{x}=\frac{R}{10^n}\left[1+\frac{1}{10^n}+\frac{1}{10^{2n}}+\frac{1}{10^{2n}}+\text{ etc.}\right]$, where R is the sequence of digits in the repetend and u the number of digits.

Call R_1 the first half of the repetend expressed in whole numbers, and R_2 the second half.

Then $R_1+R_2=999\ldots\ldots=10^{\frac{1}{2}n}-1$.

Also $R=10^{\frac{1}{2}n}R_1+R_2$, $=10^{\frac{1}{2}n}R_1-R_1+10^{\frac{1}{2}n}-1$, $=(10^{\frac{1}{2}n}-1)(R_1+1)$.

$\therefore \frac{1}{x}=\frac{(10^{\frac{1}{2}n}-1)(R_1+1)}{10^n}\cdot\frac{10^n}{10^n-1}$, or $x=\frac{10^{\frac{1}{2}n}+1}{R_1+1}$.

This gives the law of formation when the first part of the repetend is R_1.

Then, if $R_1=142$, $x=\frac{10^3+1}{143}=7$.

To find all repetends of six figures obeying the law, we proceed as follows:

$x=\frac{10^3+1}{R_1+1}=\frac{1001}{R_1+1}=\frac{13\times11\times7}{R_1+1}$. We must select such values for R_1 as will give x integral values.

If $R_1=12$, $x=77$.
If $R_1=10$, $x=9$.
If $R_1=6$, $x=143$.
If $R_1=76$, $x=13$.
If $R_1=142$, $x=7$.

If $R_1=90$, $x=11$.

Then $\frac{1}{7}=.142857\ldots\ldots$

$\frac{1}{11}=.090909\ldots\ldots$

$\frac{1}{13}=.076923\ldots\ldots$

$\frac{1}{77}=.012987\ldots\ldots$

$\frac{1}{91}=.010989\ldots\ldots$

$\frac{1}{143}=.006993\ldots\ldots$

To find eight figured repetends obeying the law, we have,

$$x=\frac{10001}{R_1+1}.$$

To find ten figured repetends we have,

$$x=\frac{100001}{R_1+1},$$

and so on. In the last instance we may take as an illustration,

$R_1=10$, then $x=9091$, and $\frac{1}{9091}=.0001099989\ldots\ldots$

The problem reduces to resolving numbers into prime factors.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

There are some very peculiar properties connected with circulating decimals, the one pointed out in the propounded question being one of them. Of all the prime numbers 2 and 5 are the only ones that do not produce circulating decimals when divided into 1. When 1 is divided by a prime number N, 2 and 5 excepted, the resulting circulator contains either $N-1$ circulating decimals or a factor of $N-1$. In the former case the period is called full. Of all the prime numbers less than 100 7, 17, 19, 23, 29, 47, 59, 61, and 97 produce full periods, the periods containing respectively 6, 16, 18, 22, 28, 46, 58, 60, and 96 circulating decimals. All these have the property that the sum of any figure in the first half of the period and the corresponding figure in the second half is equal to 9. This remarkable property may be found thus:

Let N represent the prime number, a the first half of the period, and b the second, and let the period contain $2n$ figures. We have,

$$\frac{1}{N}=\frac{a}{10^m}+\frac{a}{10^{2m}}+\frac{b}{10^{4m}}+\ldots\ldots=\left(a.10^m+b\right)\left(\frac{1}{10^{2m}}+\frac{1}{10^{4m}}+\ldots\ldots\right)$$

$$=\frac{a.10^m+b}{10^{2m}-1}. \quad \text{But } \frac{1}{N}=\frac{a}{10^m}+\frac{N-1}{N.10^m}, \text{ or}$$

$$\frac{b}{10^{2m}}+\frac{a}{10^{3m}}+\frac{b}{10^{4m}}+\ldots\ldots=\frac{N-1}{N.10^m}.$$

The first part of this equation is

$$=\frac{b.10^m+a}{10^{3m}}+\frac{b.10^m+a}{10^{5m}}+\ldots\ldots=\left(b.10^m+a\right)\left(\frac{1}{10^{3m}}+\frac{1}{10^{5m}}+\ldots\ldots\right)$$

$$=\frac{b.10^m+a}{10^m}\cdot\frac{1}{10^{2m}-1};\quad\therefore\quad\frac{b.10^m+a}{10^m}\cdot\frac{1}{10^{2m}-1}=\frac{N-1}{N.10^m},$$

$$\text{or } \frac{b.10^m+a}{10^{2m}-1}=1-\frac{1}{N}=1-\frac{a.10^m+b}{10^{2m}-1}.$$

By transposition $\frac{(a+b)10^m+(a+b)}{10^{2m}-1}=1$, or, $(a+b)(10^m+1)=10^{2m}-1$, or $a+b=10^m-1$, but the number 10^m-1 must necessarily contain m 9's only.

If the period is not a full period, but the period contains an even number of figures, the above law still holds good.

Another property of circulation is that when the number that produces the circulator is multiplied by the period, the resulting product consists of 9's only. For, designating the period by A, we have $\frac{1}{N}=A+\frac{1}{N.10^m}$.

$\therefore \frac{10^m-1}{10^m}=AN$; but 10^m-1 contains only 9's.

A third property of circulation with full periods is, that, when $1/N$ is multiplied by any any integral number from 2 to $N-1$, the resulting periods will contain the same figures and in the same succession, only beginning with a different figure. This interesting property may be found as follows :

$$\text{Let } \frac{1}{N}=\frac{a}{10}+\frac{b}{10^2}+\frac{c}{10^3}+\frac{d}{10^4}+\ldots\ldots\frac{l}{10^{N-1}}+\frac{a}{10^N}+\frac{b}{10^{N+1}}+\ldots\ldots$$

Let us, for instance, denote the sum of the remaining fractions after the third by $\frac{r}{N.10^3}$, we have $\frac{1}{N}=\frac{a}{10}+\frac{b}{10^2}+\frac{c}{10^3}+\frac{r}{N.10^3}$, whence,

$$\frac{r}{N.10^3}=\frac{1}{N}-\frac{a}{10}-\frac{b}{10^2}-\frac{c}{10^3}=\frac{d}{10^4}+\frac{e}{10^5}+\ldots\ldots\frac{l}{10^{N-1}}+\frac{a}{10^N}+\frac{b}{10^{N-1}}+\ldots$$

Multiplying by 10^3, we have,

$$\frac{r}{N}=\frac{d}{10}+\frac{e}{10^2}+\ldots\ldots\frac{l}{10^{N-4}}+\frac{a}{10^{N-3}}+\frac{b}{10^{N-2}}+\ldots\ldots,$$

which proves the proposition.

If M and N are two different prime numbers, the number of figures in the period of the circulator resulting from converting $1/MN$ into a decimal fraction is the least common multiple of the number of figures in the periods of the circulator resulting from converting $1/M$ and $1/N$ into decimal fractions. There are some more properties of circulation, but space forbids me to mention them.

NOTE. Prof. Nelson S. Roray calls attention to the error in the statement of No. 78; "minus 126" should be minus 125. The problem is found on page 344, Case II., example 4, Brooks' Higher Arithmetic. Dr. Brooks, in a letter to us, also called our attention to this error.

Profs. P. S. Berg and Chas. C. Cross sent in solutions to problem 83 too late for credit in the November number.

ALGEBRA.

76. Proposed by E. B. ESCOTT, Fellow in Mathematics, University of Chicago, Chicago, Ill.

Prove the identities

$$2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}\cdots\cdots$$

$$\frac{5-\sqrt{5}}{2}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\cdots\cdots$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let N be any number, then

$$N=\frac{Np^m}{p^m}\cdot\frac{r^m}{r^m}=\frac{p^m}{r^m}\left[1-\left(\frac{p^m-Nr^m}{p^m}\right)\right].$$

$$\therefore\sqrt[m]{N}=\frac{p}{r}\left[1-\frac{p^m-Nr^m}{p^m}\right]^{1/m}=\frac{p}{r}\left[1-\left(\frac{p^m-Nr^m}{mp^m}\right)\right.$$

$$-\frac{(m-1)}{1.2}\left(\frac{p^m-Nr^m}{mp^m}\right)^2-\frac{(m-1)(2m-)}{1.2.3}\left(\frac{p^m-Nr^m}{mp^m}\right)^3$$

$$\left.-\frac{(m-1)(2m-1)(3m--1)}{1.2.3.4}\left(\frac{p^m-Nr^m}{mp^m}\right)^4-\text{ etc.}\right].$$

Let $m=2$ and $p^m-Nr^m=1$,

$$\therefore\sqrt{N}=\frac{p}{r}\left(1-\frac{1}{p^2}\right)^{\frac{1}{2}}=\frac{p}{r}\left[1-\tfrac{1}{2}\left(\frac{1}{p^2}\right)-\frac{1}{2^3}\left(\frac{1}{p^2}\right)^2-\frac{1}{2^4}\left(\frac{1}{p^2}\right)^3\right.$$

$$\left.-\frac{5}{2^7}\left(\frac{1}{p^2}\right)^4-\text{ etc.}\right]\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1).$$

Let $m=2$ and $p^m-Nr^m=4$,

$$\therefore\sqrt{N}=\frac{p}{r}\left(1-\frac{4}{p^2}\right)^{\frac{1}{2}}=\frac{p}{r}\left[1-\tfrac{1}{2}\left(\frac{4}{p^2}\right)-\frac{1}{1.2.2^2}\left(\frac{4}{p^2}\right)^2\right.$$

$$\left.-\frac{1.3}{1.2.3.2^3}\left(\frac{4}{p^2}\right)^3-\frac{1.3.5}{1.2.3.4.2^4}\left(\frac{4}{p^2}\right)^4-\text{ etc.}\right].$$

$$\therefore\sqrt{N}=\frac{p}{r}\left[1-\frac{2}{p^2}-\frac{2}{p^4}-\frac{4}{p^6}-\frac{2.5}{p^8}-\frac{4.7}{p^{10}}-\frac{3.4.7}{p^{12}}-\text{ etc.}\right]\ldots(2).$$

In (1) let $N=2$, $p=17$, $r=12$, $\therefore p^2-2r^2=1$.

$$\therefore\sqrt{2}=\tfrac{17}{12}\left(1-\frac{1}{2.17^2}-\frac{1}{2^3.17^4}-\frac{1}{2^4.17^6}-\frac{5}{2^7.17^8}-\text{ etc.}\right).$$

$$\therefore\ 2-\sqrt{2}=2-1\tfrac{7}{12}+\frac{1}{2.12.17}+\frac{1}{2^3.12.17^3}+\frac{1}{2^4.12.17^5}+\frac{5}{2^7.12.17^7}+\text{ etc.}$$

$$2-\sqrt{2}=\tfrac{1}{2}+\tfrac{1}{1\frac{1}{2}}+\frac{1}{2.12.17}+\frac{1}{2^3.12.17^3}+\frac{1}{2^4.12.17^5}+\frac{5}{2^7.12.17^7}+\text{ etc.}$$

$$=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\left(\frac{1}{2^4.3.17.577}-\frac{1}{2^4.3.17.577.578}\right)$$

$$+\left(\frac{1}{2^4.3.17.577.578}-\frac{1}{2^4.3.17.577.578^2}\right)$$

$$+\left(\frac{1}{2^4.3.17.577.578^2}+\frac{1}{2^5.3.17.577.665857}-\right)+\text{ etc.}$$

$$\therefore\ 2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}+\frac{1}{2^5.3.17.577.665857}+\ldots\ldots$$

In (2) let $N=5$, $p=7$, $r=3$, $\therefore\ p^2-5r^2=4$.

$$\therefore\ \sqrt{5}=\tfrac{7}{3}\left(1-\frac{2}{7^2}-\frac{2}{7^4}-\frac{4}{7^6}-\frac{2.5}{7^8}-\frac{4.7}{7^{10}}-\text{ etc.}\right)$$

$$\frac{5}{\sqrt{2}}=\tfrac{6}{7}-\frac{1}{3.7}-\frac{1}{3.7^3}-\frac{2}{3.7^5}-\frac{5}{3.7^7}-\frac{2}{3.7^8}-\text{ etc.}$$

$$\frac{5-\sqrt{5}}{2}=\tfrac{5}{2}-\tfrac{7}{6}+\frac{1}{3.7}+\frac{1}{3.7^3}+\frac{2}{3.7^5}+\frac{5}{3.7^7}+\frac{2}{3.7^8}+\text{ etc.}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}$$

$$+\left(\frac{1}{3.7.47}-\frac{2}{3.7.47.49}\right)+\left(\frac{2}{3.7.47.49}-\frac{4}{3.7.47.49^2}\right)$$

$$+\left(\frac{4}{3.7.47.49^2}+\frac{1}{3.7.47.2207}-\right)+\text{ etc.}$$

$$\therefore\ \frac{5-\sqrt{5}}{2}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\ldots\ldots$$

The above formulæ were used by Dr. Artemas Martin for extracting the root of numbers several years ago.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Since $2-\sqrt{2}$ is rather more than $\frac{1}{2}$, we can, confining ourselves to 4 terms, put $2-\sqrt{2}=\frac{1}{2}+\frac{1}{2^2x}+\frac{1}{2^3xy}+\frac{1}{2^4xyz}$, or by transposition, $\sqrt{2}=\frac{3}{2}-\frac{1}{4x}-\frac{1}{8xy}-\frac{1}{16xyz}$. Squaring and omitting the term $\frac{1}{256x^2y^2z^2}$, we get, after multiplying by 64 x,

$$16x=48-\frac{4}{x}+\frac{24}{y}-\frac{1}{xy^2}+\frac{12}{yz}-\frac{4}{xy}-\frac{2}{xyz}-\frac{1}{xy^2z},$$

whence $x=3$, and substituting this value and multiplying by $3y$, we get

$$4y=68-\frac{1}{y}+\frac{34}{z}-\frac{1}{yz};$$

$\therefore$ $y=17$, and substituting it, and multiplying by $17z$, we get $z=578-1=577$. If another term had been desired we would have annexed to the above series the term $\frac{1}{2^5xyzt}$ and proceeded in the same manner, though for 4 terms, the series is sufficiently convergent to render the value of $2-\sqrt{2}$ correct for at least 8 decimals.

Since $\frac{5-\sqrt{5}}{2}$ is $=1+$ fraction, we put, restricting the series to 4 terms besides 1,

$$\frac{5-\sqrt{5}}{2}=1+\frac{1}{x}+\frac{1}{xy}+\frac{1}{xyz}+\frac{1}{xyzt},$$

$$\text{whence } \sqrt{5}=3-\frac{2}{x}-\frac{2}{xy}-\frac{2}{xyz}-\frac{2}{xyzt}.$$

Squaring and omitting the term $4/x^2y^2z^2t^2$, we have, transposing, suppressing the common factor 4, and multiplying by x,

$$x=3-1/x+3/y-1/xy^2-1/xy^2z^2+3/yz+3/yzt$$
$$-2/xy-2/xyz-2/xyzt-2/xy^2z-2/xy^2z^2t, \text{ whence } x=3.$$

Substituting this value, and multiplying by $3y$, we have

$$y=7-1/y-1/yz^2+7/z+7/zt-2/yz-2/yz^2t; \text{ whence } y=7.$$

Substituting, and multiplying by $7z$, we get

$$z=47-1/z+47/t-2/zt; \text{ whence } z=47.$$

Substituting, and multiplying by $47t$, we get $t=2207$.

Such series converge very rapidly. In German works they are called "Theilbruchreihen," signifying *partial fraction series*. Every common fraction may be converted into such a series by the following process.

Let the fraction be $\frac{15}{19}$.

$\therefore$ $\frac{15}{19}=\frac{1}{2}+1/2.2+1/2.2.7+1/2.2.7.10+1/2.2.7.10.19,$

so that, taking successively 1, 2, 3, 4 of these, we get as in continuous fractions, approximate values, as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{11}{14}$, $\frac{223}{285}$. As in continuous fractions, the numerator of the difference of any two consecutive approximate fractions is always $=1$. Supposing the indeterminate equation $15x-19y=1$. Changing $\frac{15}{19}$ by the above method into such a

$$\begin{array}{r} 15)19(2 \\ \hline 11)19(2 \\ \hline 3)19(7 \\ \hline 2)19(10 \\ \hline 1)19(19 \end{array}$$

series, and take the last approximate fraction, viz, $\frac{221}{280}$, $x=280$, $y=221$, will furnish two values; also, the preceding fraction $\frac{11}{14}$, $x=14$, $y=11$.

77. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, N. H.

Solve the equation, $(6x^2+x-3)^2-48^2=(x+15)^2$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and A. H. BELL, Hillsboro, Ill.

$(6x^2+x-3)^2-(48)^2=(x+15)^2$.

$\therefore\ 3x^4+x^3-3x^2-3x-210=0$.

$\therefore\ x_1=2.924412+$,

$x_2=-3.041623-$,

$x_3=2.804395\sqrt{-1}-.158044$.

$x_4=-2.804395\sqrt{-1}-.158044$.

II. Solution by J. MARCUS BOORMAN, Consultative Mechanician, Counsellor at Law, Inventor, Etc., Hewlett, Long Island, New York.

The *real* roots of $x^4+\frac{1}{3}x^3-x^2-x=70$ (the forms to which the given equation reduces) are:

$x=+2.924412\ 149966\ 623189\ 6108(58211\ )$ true to "(" marks,—*probably*
$x_1=-3.041622\ 694570\ 750484\ 61819(75892\ )$ 25 decimals true.

	x^4 +	$\frac{1}{3}x^3$	$-1x^2$	$-1x$	-70
Found thus: At sight $x=\pm3$(near), try -3.04			$+8.24-$	-22.01	69.9504
	$-2.71-\times3$		$+7.24-$	-23.01	$0.04-$
Multiply by $-3.$ $-.04$	$+.13$		21.72	69.03	error $+$*
	$.1084+$		$29-$	$.9204$	
	$+8.24-$		-22.01	69.9504	

*The root $\therefore$ is $-2.0316+$ nearly.

Like treatment by $+2.9$ gives 2.925 near true. I get the rest in a more concise way (shorter than Horner's).

The *resulting* equations of above roots are: (the coëfficients appear on *face of computation*)

$0=x^3+(+\text{root}+\frac{1}{3})x^2+8.526990\ 472861\ 28178\text{etc.}\ x+23.936434,541435,17396+.$

$0=x^3+(-\text{root}+\frac{1}{3})x^2+7.237594\ 384604\ 24939\text{etc.}\ x-23.914031,334310,10968+.$

$0=(5.966034,\ 84\text{etc})x^2+(1.289496,088257,03238+)x+46.950465,875795,28364+.$

Hence $x^2+0.216122,\ 788722x=-7.869626,\ 49385$ etc.

$\therefore\ x_1=-0.108$ etc. $\pm\sqrt{-7.890+}$.

Shorter and *more accurate* than Horner's method, *especially* for 5th degree and 6th degree equations.

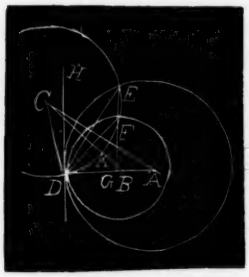
The other two roots are imaginary.

GEOMETRY.

80. Proposed by J. C. GREGG, Superintendent of Schools, Brazil, Ind.

One circle touches another internally, and a third circle whose radius is a mean proportional between their radii, passes through the point of contact. Prove that the other intersections of the third circle with the first two are in a line parallel to the common tangent of the first two. [From *Phillips and Fisher's Geometry.*]

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and the PROPOSER.



In the two triangles CDB and CDA, the angle CDG is common; also $DB : DC = DC : DA$.

$\therefore$ The two angles are similar and $\angle BCD = \angle CAD$, $\angle CBD = \angle ACD$. But $\angle DCB = \angle DEG$, both being measured by arc DK.

$\therefore \angle DEG = \angle CAD$.

Since DE is perpendicular to CA, EG must be perpendicular to CA.

$\therefore$ HD and EG are parallel.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa, and J. SCHEFFER, A. M., Hagerstown, Md.

Taking the common tangent of the first two circles as the axis of x and the common diameter as the axis of y and supposing the tangent of the third circle, through the origin to make the angle α with the axis of x, the equations of the circles are,

$$x^2+y^2-2ay=0,$$

$$x^2+y^2-2by=0, \text{ and}$$

$$x^2+y^2-2c(x\sin\alpha+y\cos\alpha)=0.$$

If the first and third circles intersect,

$$y=0 \text{ or } \frac{2ac^2\sin^2\alpha}{a^2+2ac\cos\alpha+c^2}.$$

If the second and third circles intersect,

$$y=0 \text{ or } \frac{2bc^2\sin^2\alpha}{b^2+2bc\cos\alpha+c^2}.$$

If the line through the second intersections is parallel to the axis of x the two values of y are equal.

$\therefore$ $ab^2+ac^2=a^2b+bc^2$. Whence $c^2=ab$.

If $c=\sqrt{ab}$, the distance y of the intersections from the common tangent is in both cases

$$\frac{2ab\sin^2\alpha}{a+\sqrt{ab}\cos\alpha+ab}.$$ Q. E. D.

Also solved by *COOPER D. SCHMITT* and *CHARLES C. CROSS.*

81. Proposed by CHAS. C. CROSS, Laytonsville, Md.

A circle is drawn bisecting the lines joining the points of contact of the escribed circles with the sides produced. Another circle is drawn passing through the centers of the circles drawn tangent externally to the in-circle and internally to the sides of the triangle. Prove that the centers of these two circles, the incenter and the circumcenter are collinear.

This problem is reprinted to correct an error in its enunciation. Inscribed is changed to *escribed*. EDITOR.

82. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, Cal.

If the extremities of the base of a triangle be joined by straight lines to the exterior angles of squares constructed upon its two sides, the superior pair of lines thus drawn intersect at right angles; the inferior pair intersect at a point in a line drawn from the vertical angle perpendicular to the base.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.; COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tenn.; and CHAS. C. CROSS, Laytonsville, Md.

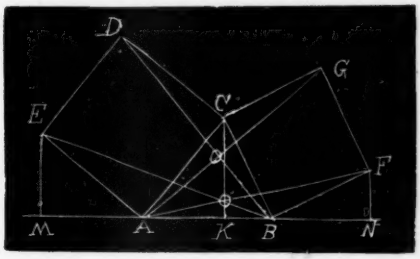
1. In the two triangles BCD and ACG, we have $CG=CB$, $CD=CA$, $\angle BCD=\angle ACG$.

$\therefore \triangle BCD=\triangle ACG$ and $\angle BDC=\angle CAG$.

Hence, in quadrilateral $EAOD$, $\angle EAO+\angle EDO$=two right angles.

$\therefore \angle AED+\angle AOD$=two right angles, but $\angle AED$=a right angle and therefore AOD is a right angle.

$\therefore AG$ and BD are perpendicular to each other.



2. Let $AK=d$, $CK=f$. Then $BN=CK=f$, $FN=KB=b-d$, $AM=CK=f$, $EM=AK=d$.

$\therefore$ Equation to AF is $y=\frac{b-d}{b+f}x$, equation to BE is $y=-\frac{d}{b+f}(x-b)$.

These lines intersect at a distance $x=d$.

$\therefore AF$ and BE intersect on CK, the perpendicular from C on AB.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa; J. SCHEFFER, A. M., Hagerstown, Md.; and J. C. GREGG, Superintendent of Schools, Brazil, Ind.

Let $ACDE$ represent the square erected on the side AC, and $BCGF$ that on BC. Let O be the point of intersection of BD and AG, and O' that of BE and AF. In the triangles ACG and BCD, we have $CG=BC$, $AC=CD$, $\angle ACG=\angle BCD$. $\therefore \triangle ACG=\triangle BCD$. $\therefore \angle CDB=\angle CAG$, consequently the points A, O, C, D, are concyclic, and since $\angle DCA$ is a right angle, DOA must be a right angle.

Let EM and FN be the perpendicular let fall from E and F, respectively, upon AB produced. $O'H$ the perpendicular let fall from O' upon AB, and H' the foot of the perpendicular from C upon AB. We have

$$EM : O'H = MB : BH, \text{ and}$$
$$O'H : FN = AH : AN.$$
$$\therefore EM : FN = MB \times AH : AN \times BH,$$

but $MB = AM + AB = CH' + AB$, and $AN = BN + AB = CH' + AB$.

$\therefore MB = NA$, $\therefore EM : FN = AH : BH$, but $EM = AH'$, $NF = BH'$.

$\therefore AH' : BH' = AH : BH$.

$\therefore AH = AH'$, $BH = BH'$. Q. E. D.

CALCULUS.

63. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

What is the volume removed by boring an auger hole radius r through a right cylinder radius R, the center of the auger hole to pass at a distance c from the axis of the cylinder and inclined to the axis at an angle α?

I. Solution by the PROPOSER.

Let the axis of the cylinder whose radius is R coincide with the y-axis and let the axis of the cylinder whose radius is r intersect the z-axis at a distance c from the xy-plane, being parallel to the xy-plane and making an angle α with the y-axis. Pass a plane parallel to the xy-plane through the solid common to the two cylinders and at a distance z from the origin of coördinates. The intersection of this plane with the surface of the two cylinders forms a parallelogram, whose length is $2\sqrt{R^2-z^2}$ and whose width is $2\csc\alpha\sqrt{r^2-(z-c)^2}$. Hence its area is

$$\csc\alpha\sqrt{[r^2-(z-c)^2][R^2-z^2]}.$$

$\therefore V = \int_{c-r}^{c+r} \csc\alpha\sqrt{(R^2-z^2)[r^2-(z-c)^2]}\,dz$. Let $y = \frac{p+qy}{1+y}$, p and q to be determined from the conditions that the odd powers of z in the expansion under the radical shall vanish. From this condition we find $pq = R^2$ and

$$p+q = \frac{R^2-r^2+c^2}{c} \text{ or } \frac{R^2+r^2-c^2}{r}$$

according as $c > r$ or $< r$. From these two equations we find $p =$

$$\frac{R^2-r^2+c^2+\sqrt{(R^2-r^2+c^2)^2-4R^2c^2}}{2c} \text{ or } \frac{R^2+r^2-c^2+\sqrt{(R^2+r^2-c^2)^2-4R^2r^2}}{2r}$$

and q has the conjugate value of p. Making the substitution of $(p+qy)/(1+y)$, the values of p and q being replaced by their values in terms of R, r, and c as found above. The expression for the volume reduces to the following form :

$$V=-\csc\alpha\frac{\sqrt{(R^2-r^2+c^2)-4R^2c^2}}{c}[(R^2+c^2-r^2)^2$$

$$+(R^2+c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}-4R^2c^2]$$

$$[(R^2-c^2)^2-2c^2(R^2+r^2)+c^4-(R^2-c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}$$

$$\times\int\Big[\Big(1+\frac{(R^2+c^2-r^2)^2-(R^2+c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}-4R^2c^2}{(R^2+c^2-r^2)^2+(R^2+c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}-4R^2c^2}y^2\Big)$$

$$\Big(1+\frac{(R^2-c^2)^2-2c^2(R^2+r^2)+c^4-(R^2-c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}}{(R^2-c^2)^2-2c^2(R^2+r^2)+c^4+(R^2-c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}}y^2\Big)\Big]^{\frac{1}{2}}$$

$$\times\frac{dy}{(1+y)^4},$$

which is of the form $k\int\sqrt{(1+my^2)(1+ny^2)}\,\frac{dy}{(1+y)^4}$. Let $y=x/\sqrt{-m}$.

Then $V=-mk\sqrt{-m}\int\sqrt{[1-x^2][1-(n/m)x^2]}\,\frac{dx}{(x+\sqrt{-m})^4}$

$$=-m\sqrt{-m}\,k\int\frac{[1-x^2][1-(n/m)x^2]dx}{[x+\sqrt{-m}]^4\sqrt{[1-x^2][1-(n/m)x^2]}}$$

$$=-m\sqrt{-m}\,k\int\frac{\{1-[(m+n)/n]x^2+(n/m)x^4\}/[x+\sqrt{-m}]^4}{\sqrt{[1-x^2][1-(n/m)x^2]}}dx.$$

By multiplying both numerator and denominator of the numerator by

$$(x^4-mx^2+m^2)-\sqrt{-m}\,x(x^2-m),$$

we reduce the numerator to a function of x^2 only. From this point by some labor we can obtain the integral in terms of logarithmic and circular, and elliptic functions of the first, second, and third kinds. We have made this integration but the result is too long and complicated to put in print.

When $c<r$, c and r change places in the formula derived by the above solution.

COROLLARY 1. If $c=0$, and $\alpha=\frac{1}{2}\pi$, $V=\int_0^r\sqrt{(r^2-z^2)(R^2-x^2)}dz$.

Let $z=rx$, and $r/R=e$.

Then $V=2Rr^2\int_0^1\sqrt{(1-x^2)(1-e^2x^2)}dx$.

By following out the method indicated above we get

$$V=\tfrac{8}{3}R^3[(1+e^2)E(e)-(1-e^2)F(e)].$$

COROLLARY 2. By Prof. Henry Heaton, Atlantic, Iowa.

If $R=r$, $V=1/\sin\alpha\int_0^{R-\frac{1}{2}c}\sqrt{R^2-(x-\frac{1}{2}c)^2}\sqrt{R^2-(x+\frac{1}{2}c)^2}\,dx$, where $x+\frac{1}{2}c$ is the distance from axis of cylinder to the plane.

Put $x=(R-\frac{1}{2}c)\sin\theta$, and $(R-\frac{1}{2}c)/(R+\frac{1}{2}c)=e$.

Then $V=(2/\sin\alpha)(R-\frac{1}{2}c)^2(R+\frac{1}{2}c)\int_0^{\frac{1}{2}\pi}\cos^2\theta\sqrt{1-e^2\sin^2\theta}\,d\theta$

$$=(2/\sin\alpha)(R+\tfrac{1}{2}c)^3[(1+e^2)E(e,\tfrac{1}{2}\pi)-(1-e^2)F(e,\tfrac{1}{2}\pi)].$$

COROLLARY 3. By Prof. G. B. M. Zerr, Lebanon, Va., and Prof. C. W. M. Black, Wilbraham, Mass.

If $R=r$ and $c=0$, $V=(16r^3)/(3\sin\alpha)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let $x^2+z^2=R^2$, be the equation to the cylinder.

Then $(x\cos\alpha+y\sin\alpha)^2+(z-c)^2=R^2$, is the equation to the auger hole.

The limits of y are,

$$y=\frac{\sqrt{R^2-(z-c)^2}-x\cos\alpha}{\sin\alpha}\text{ and }y=-\frac{\sqrt{R^2-(z-c)^2}+x\cos\alpha}{\sin\alpha}.$$

$$\therefore\ V=(4/\sin\alpha)\int_0^R\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{R^2-(z-c)^2}\,dx\,dz$$

$$=(2/\sin\alpha)\int_0^R\Big[\Big(\sqrt{R^2-x^2}-c\Big)\sqrt{R^2-(\sqrt{R^2-x^2}-c)^2}$$

$$+R^2\sin^{-1}\Big(\frac{\sqrt{R^2-x^2}-c}{R}\Big)+\Big(\sqrt{R^2-x^2}+c\Big)\sqrt{R^2-(\sqrt{R^2-x^2}+c)^2}$$

$$+R^2\sin^{-1}\Big(\frac{\sqrt{R^2-x^2}+c}{R}\Big)\Big]dx.$$

This does not appear to be easy to reduce.

64. Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics in Cleveland West High School, Berea, Ohio.

Find the volume and surface generated by revolving about Y, the catenary $y=\frac{1}{2}a(e^{x/a}+e^{-x/a})$, from $x=0$ to $x=a$.

[*Osborne's Calculus*, page 255, example 8].

I. Solution by S. ELMER SLOCUM, Professor of Mathematics Union College, Schenectady, New York; J. M. BANDY, A. M., Professor of Mathematics in Trinity College, Trinity, N. C.; COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tenn.; and B. F. SINE, Principal of Rock Enon High School, Rock Enon Springs, Va.

$$V=\pi\int_o^a x^2\,dy. \quad y=\tfrac{1}{2}a(e^{x/a}+e^{-x/a}). \quad \therefore\ dy=\tfrac{1}{2}(e^{x/a}-e^{-x/a})dx.$$

$$\therefore\ V=\tfrac{1}{2}\pi\int_o^a x^2(e^{x/a}-e^{-x/a})dx.$$

Integrating by parts,

$$V=\tfrac{1}{2}\pi\Big[(ax^2e^{x/a}-2a^2xe^{x/a}+2a^3e^{x/a})+(ax^2e^{-x/a}+2a^2xe^{-x/a}+2a^3e^{-x/a})\Big]_o^a.$$

$$\therefore\ V=\tfrac{1}{2}\pi[a^3e+5a^3e^{-1}-4a^3]=\tfrac{1}{2}\pi a^3[e+5e^{-1}-4].$$

$$S=2\pi\int x\,ds=2\pi\Big[xs-\int s\,dx\Big],$$

where s=length of arc of catenary, $=\tfrac{1}{2}a(e^{x/a}-e^{-x/a})$.

$$\therefore\ S=2\pi\Big[\tfrac{1}{2}ax(e^{x/a}-e^{-x/a})-\tfrac{1}{2}a\int(e^{x/a}-e^{-x/a})dx\Big]\Big/_o^a$$

$$\text{or } S=2\pi\Big[\tfrac{1}{2}ax(e^{x/a}-e^{-x/a})-\tfrac{1}{2}a^2(e^{x/a}+e^{-x/a})\Big]\Big/_o^a.$$

$$\therefore\ S=2\pi[\tfrac{1}{2}a^2(e-1/e)-\tfrac{1}{2}a^2(e+1/e)+\tfrac{1}{2}2a^2]. \quad S=2\pi a^2(1-e^{-1}).$$

II. Solution by J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tenn.; and HENRY HEATON, M. Sc., Atlantic, Iowa.

(a). The volume is given by the equation

$$V=\pi\int_o^a x^2\,dx=\tfrac{1}{2}\pi\int_o^a x^2(e^{x/a}-e^{-x/a})dx.$$

Let $x=ay$, then the limits of y are 0 and 1.

Hence $V=\pi\int_o^a x^2\,dy=\tfrac{1}{2}\pi a^3\int_0^1 y^2(e^y-e^{-y})dy.$

Integrating by parts, we find,

$$\int y^2e^y\,dy=(y^2-2y+2)e^y, \text{ and } \int y^2e^{-y}dy=-(y^2+2y+2(e^{-y}.$$

$$\text{Therefore } V=\frac{\pi a^3}{2}\Big[(y^2-2y+2)e^y+(y^2+2y+2)e^{-y}\Big]_0^1$$

$$=(\pi a^3/2)(e+5e^{-1}-4).$$

(b). The surface is given by

$$S=2\pi\int_o^a x\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx=\pi\int_o^a x(e^{x/a}+e^{-x/a})dx$$

$$=\pi a\Big[e^{x/a}(x-a)-e^{-x/a}(x+a)\Big]_o^a=2\pi a^2(1-e^{-1}).$$

MECHANICS.

54. Proposed by C. H. WILSON, Poughkeepsie, N. Y.

A body slides from rest down a series of smooth inclined planes, whose total heights are h feet. Show that the velocity at the bottom is $\sqrt{(2gh)}$ feet per second. [From *Wright's Mechanics.*]

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let $v_1, v_2, v_3, \ldots\ldots v_n$ be the velocities at the bottoms of the planes, $h_1, h_2, h_3, \ldots\ldots h_n$ their respective heights.

$$\therefore v_1=\sqrt{2gh_1}, \quad v_2=\sqrt{2gh_1+2gh_2}=\sqrt{v_1^2+2gh_2},$$

$$v_3=\sqrt{v_2^2+2gh_3}=\sqrt{2gh_1+2gh_2+2gh_3},$$

$$v_n=\sqrt{2gh_1+2gh_2+\ldots\ldots+2gh_n}=\sqrt{2g(h_1+h_2+h_3+\ldots\ldots+h_n}=\sqrt{2gh},$$

since $h_1+h_2+h_3+\ldots\ldots+h_n=h$.

Also solved by *HENRY HEATON, C. W. M. BLACK,* and *CHAS. C. CROSS.*

55. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Miss.

Three equal heavy spheres, each of weight W, are placed on a rough ground just not touching each other. A fourth sphere of weight nW is placed on the top touching all three. Show that there is equilibrium if the coefficient of friction between two spheres is greater than $\tan\frac{1}{2}\alpha$, and that between a sphere and the ground is greater than $\tan\frac{1}{2}\alpha n/(n+3)$, where α is the inclination to the vertical of the straight line joining the centers of the upper and one lower sphere.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, O.

Let α be the angle which the line of centers of the upper sphere and each of the lower makes with the vertical, R the reaction of the upper and each of the lower spheres, m the corresponding coefficient of friction, m_1 the coefficient for each lower sphere and the plane.

The system is kept at rest by the weights nW, W, acting vertically, R the reaction in direction of centers, mR, friction acting in the tangent through the point of contact of the upper and lower spheres, and $m_1(nW+3W)$ horizontally and inward.

For the equilibrium of the upper sphere, resolving vertically,

$$3R\cos\alpha+3Rm\sin\alpha=W \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and for the lower, resolving horizontally,

$$\tfrac{1}{3}(nW+3W)m_1=R\sin\alpha-Rm\cos\alpha \ldots\ldots\ldots\ldots\ldots (2).$$

Also, $R=\frac{1}{3}nW$ ..... (3).

(1), (2), and (3) gives $m=\tan\frac{1}{2}\alpha$, $m_1=[n/(n+3)]\tan\frac{1}{2}\alpha$.

Also solved by *G. B. M. ZERR* and the *PROPOSER.* Their solutions will appear in next number.

DIOPHANTINE ANALYSIS.

55. Proposed by O. W. ANTHONY, M. Sc. Instructor in Mathematics in Boys' High School, New York City.

Construct a general Magic Square whose sum is $3m$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

$m+bn$	$m-(a+b)n$	$m+an$
$m+(a-b)n$	m	$m-(a-b)n$
$m-an$	$m+(a+b)n$	$m-bn$

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

I assume that the method of constructing Magic Squares is understood, and that they are made up of numbers in arithmetical progression.

Let p be the first term, q the common difference, and n the number of rows.

Then n^2=number of cells and of the numbers used; the sum of the series divided by n gives the sum of the numbers in each row=$3m$.

Or $\frac{n^2[2(n^2-1)+2p]}{2n}=3m$, or, $n[q(n^2-1)+2p)=6m$ ....(1).

As there are three unknown quantities, two of them may be assumed, and there will be a result for each assumption of both numbers.

Then $n=3$ and reducing we have $4q+p=m$; or $p=m-4q$, in which m may be any multiple of q greater than $4q$.

Take $m=5q$, then $q=m/5$, and $p=q$. In this m may be any number divisible by 5, and for every value we have a magic square, whose sum is $3m$.

Again, take $m=6q$, then $p=2q$, and $q=m/6$; m may be any number divisible by 6; and for every value we have a magic square, whose sum is $3m$.

In the same manner, we find that m may be any number divisible by any one of the numbers in the natural series from 5 upwards.

Now take $n=5$; substitute in (1), and we have by reducing $120q+10p=6m$. To simplify assume $m=5t$, and we have $12q+p=3t$; or $p=3t-12q$, in which t may be any number greater than $4q$.

Take $t=5q$; then $m=25q$, and $p=3q$, and $q=m/25$. So m may be any number divisible by 25.

In same manner, take $t=6q$, $7q$, etc., and we have $q=m/30$, $m/35$, etc., so that we may have m=any number divisible by 5 times any one of the natural numbers from 5 upwards, and as many magic squares of 5 rows.

In the same manner, we may obtain similar results by taking n=any other odd number.

Or, we can obtain a general solution directly from (1). Take $m=nt$, $n=2s+1$, substitute and reduce and we find $p=3t-2q(s^2+s)$, in which t may be $=q(s^2+s)$, or any multiple of it. Take $t=q(s^2+s)$; and then $p=q(s^2+s)$, (the first term); and $m=nt=q(2s+1)(s^2+s)$, and $q=m/[s(s+1)(2s+1)]$ (the common difference); in which s may be any integral, and $m=s(s+1)(2s+1)$, or any multiple of it.

But this formula does not enable us to obtain least values of p, q, and m, as n varies.

56. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

If $\phi(R)$ is the number of integers which are less than R and prime to it, and if y is prime to R, show that $y^{\phi(R)}-1\equiv 0(\mathrm{mod}.R)$.

Solution by the PROPOSER, and J. O. MAHONEY, B. E., M. Sc., Lynnville, Tenn.

Let $1, m, n, p, \ldots\ldots(R-1)$ denote the $\phi(R)$ numbers less than R and prime to it ; now y can be any one of those numbers.

$\therefore$ $y, my, ny, py, \ldots\ldots(R-1)y$ are all prime to R and all different.

There are $\phi(R)$ of such products and since when these products are divided by R the remainders are all prime to R and all different, the $\phi(R)$ remainders must be $1, m, n, p, \ldots\ldots(R-1)$ though not necessarily in this order.

$\therefore$ $y.my.ny.py\ldots\ldots (R-1)y$ must differ from $1.m.n.p\ldots\ldots (R-1)$ by a multiple of R.

$\therefore$ $\{y^{\phi(R)}-1\}mnp\ldots\ldots(R-1)=$a multiple of R.

But $mnp\ldots\ldots(R-1)$ is prime to R.

$\therefore$ $y^{\phi(R)}-1\equiv 0(\mathrm{mod}.R)$.

57. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Each of *five* of the digits may be the terminal figure of a perfect integral square. Each of *eighteen* combinations of two digits may be the *two* terminal figures of an integral square. Each of *one hundred and nineteen* combinations of three digits may be the *three* terminal figures of an integral square. *Under these conditions*, what is the greatest number of arrangements of the nine digits, all taken together, whose three terminal figures shall be those of a square number ?

No solution of this difficult problem has been received. Can any of our readers furnish the desired solution ? EDITOR.

MISCELLANEOUS.

53. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.

(a) What is the highest north latitude in which the Sun will shine in at the north window of a building at least once in a year ?

(b) How many days will it shine in at the north window of a building in latitude 41° N. ?

Note by SAMUEL HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

Whenever the Sun, or any part of it, is north of the prime vertical, it must then shine on the north side of buildings. From the time of vernal equinox, to the autumnal equinox, the Sun will be north of the prime vertical during some part of every day, and will shine on the north side of buildings some part of *every* day for about half a year, and in *all* latitudes north of the equator. Hence the answer for (a) is 90° N. latitude, and for (b) 186 days, but if the Sun's upper

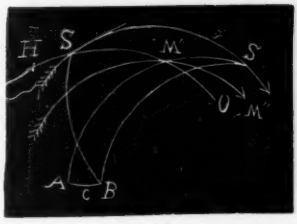
limb, and refraction, be considered the days will be 187 or 188. The answer for 41° N. is *good* for *any* other latitude north, while the problem seems to imply that an answer for 41° is different for other latitudes.

54. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

On latitude 40° N.$=\lambda$, when the Moon's declination is 5° 23′ N.$=\delta$, and the Sun's 9° 52′ S.$=-\delta$, how long after sunset will the two horns or cusps of the Moon's crescent (recently new) set at the same moment, the crescent with its back *down* having touched the horizon first? Semi-diameters, refraction, and parallax not considered.

I. Solution by the PROPOSER.

Let B be the celestial north pole, A the zenith, AB an arc of the meridian equal the co-latitude$=c=50°$, HO a portion of the horizon, SS' and MM'' portions of the diurnal arcs of the Sun and Moon, the Sun setting at S, and the Moon at M'; BS=the polar distance of the Sun$=BS'$, and BM' the polar distance of the Moon, and AM' the zenith distance of the Moon$=90°$.



Produce the vertical circle AM' to S', S' being the place of the Sun when the Moon sets at M'. The line joining the Moon's cusps must be at right angles to the line $M'S'$ joining the centers of the Sun and Moon, and as the horison is at right angles to $AM'S'$, the line of the cusps must lie on the horizon and set when the Moon's center sets. Put $\angle ABS=\phi$=Sun's hour angle when it sets, and $\angle ABS'=\theta$= Sun's hour angle when the Moon sets, and $\angle ABM'=\psi$=Moon's hour angle when it sets.

Then we have $\cos\phi=\tan\delta'\tan\lambda$. $\therefore\ \phi=81°\ 36'\ 29''$, and $\cos\psi=-\tan\delta\ \tan\lambda$. $\therefore\ \psi=94°\ 32'\ 7''$. Take an auxiliary arc χ', and $\tan\chi'=\cos\psi\cot\delta$. $\therefore\ \chi'=40°\ 0'\ 1''$, then $\cot A=\sin(c-\chi')\cot\psi\operatorname{cosec}\chi'$. $\therefore\ A=82°\ 57'\ 55''$. Take an auxiliary angle γ', and $\cot\gamma'=\tan A\sin\lambda$. $\therefore\ \gamma'=10°\ 52'\ 2''$. Then $\cos\gamma'\cot\lambda\tan-\delta'=-\cos\gamma$. $\therefore\ \gamma=101°\ 44'\ 43''$, and $\angle ABS'=\gamma'+\gamma=\theta=112°\ 36'\ 45''$, and $\theta-\phi=31°\ 0'\ 16''=2$ hours, 4 minutes, 1 second.

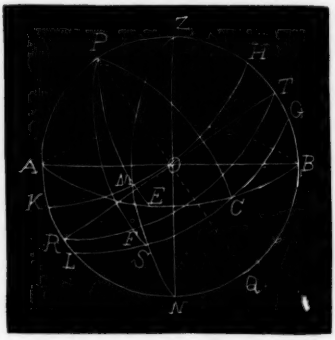
NOTE. The synchronous setting or rising of the cusps of a crescent Moon, is a phenomenon which must occur frequently in the tropics, and rarely or not at all beyond latitude 45°. On the 4th of July, 1897, such a moonset was very nearly accomplished, and another, almost perfect, will occur February 22, 1898, the declinations being then as given in the problem. Few persons in the northern states have ever seen the Moon set with both horns vertical.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let O be the observer, Z his zenith, HMK Moon's path, $GCSL$ Sun's path, $TEFR$ celestial equator, $AMCB$ the horizon. Let M be the position of the Moon when setting. Then, in order that the horns may set at the same time, S, M, where S is the Sun, must be on the same meridian, $ZMSN$.

$AP=\lambda=40°$. $ME=\delta=5°\ 23'$ N. $SF=\delta_1=9°\ 52'$ S. In the triangle

PMZ, $PZ=90^\circ-\lambda_1$, $ZM=90^\circ$, $PM=90^\circ-\delta$. Let $\angle ZPM=h$, $\angle PZM=z$, $ZPC=h_1$, $\angle ZPS=\beta$.



Then $\cos h=-\tan\lambda\tan\delta=-\tan 40^\circ\tan 5^\circ 23'$. $\therefore h=94^\circ\ 32'\ 6.8''$.

$\sin z=\cos\delta\sin h$. $\therefore z=82^\circ\ 57'\ 54''$.

$\cos h_1=-\tan\lambda\tan\delta_1$. $\therefore h_1=81^\circ\ 36^\circ\ 29$.

In triangle ZPS, $PS=90^\circ+\delta_1$, $PZ=90^\circ-\lambda$. $\angle PZS=z$.

$\sin PSZ=(\sin z\cos\lambda)/\cos\delta_1$. $\therefore PSZ=\lambda_1=50^\circ\ 30'\ 22''$.

$$\cot\tfrac{1}{2}\beta=\frac{\sin\tfrac{1}{2}(180^\circ-\lambda+\delta_1)}{\sin\tfrac{1}{2}(\lambda+\delta_1)}\tan\tfrac{1}{2}(z-\lambda_1).$$

$\therefore \beta=112^\circ\ 36'\ 47''$.

β=Sun's hour angle at moonset ; h_1= Sun's hour angle at sunset ; $\beta-h_1=31^\circ\ 0'\ 18''$=2 hours, 4 minutes, 1.2 seconds after sunset.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

89. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, N. J.

Solve by pure arithmetic. A criminal having escaped from prison traveled 10 hours before his escape was known; he was then pursued so as to be gained upon 3 miles an hour; after his pursuers had traveled 8 hours they met an express going at same rate as themselves, who had met the criminal 2 hours and 24 minutes before; in what time from the commencement of the pursuit will they overtake him?

90. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

A owes \$6000 which is drawing 6% interest. He wishes to pay off the debt in six equal annual payments, the first to be due in one year. The whole portion of the claim unpaid at the end of each year to be accounted as principal, and to draw interest to the time of the next payment. Required the amount of each payment, so the six equal payments will discharge the obligation, interest and all.

91. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.

\$1000.00. Cleveland, Ohio, May 26, 1893.

Two years after date I promise to pay John Davis, or order, one thousand dollars, for value received, interest six per cent. payable annually. J. M. LEWIS.

Indorsements: December 14, 1895, \$560.56; May 11, 1896, \$10.02; June 14, 1697, \$545.06.

Find, by the United States' Rule, the amount due August 2, 1897.

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 1.

GEOMETRY.

86. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, O.

Prove that the four conics which have S for focus and which touch the three sides of each of the triangles ABC, AEF, BFD, CDE, have their lata-recta equal.

87. Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military Academy, Washington, Miss.

Given any two straight lines in space, AB, CD, which do not intersect. So construct upon one of the lines as base, a triangle, having its vertex in the other line, such that its perimeter shall be a minimum.

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 1.

CALCULUS.

68. Proposed by EDWARD DRAKE ROE, Jr., A. M., Associate Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If $a^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}$ to r steps be denoted by $a^{\overset{r}{x}}$, and if $y=x^{\overset{r}{x}}$, prove that

$$D_x y = x^{x^{r}+x^{r-1}+\ldots+x^{1}}(\log x)^{r-1}(1+\log x)+\sum_{k=2}^{k=r} x^{x^{r}+x^{r-1}+\ldots+x^{k-1-1}}(\log x)^{r-k}.$$

69. Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics, State University, Eugene, Ore.

An elliptic fence encloses a field whose major and minor axes are $2a$ and $2b$, respectively. The ends of a rope, the length of which is equal to the length of the fence, are fastened outside the fence and at the extremities of the major axis. A horse is tethered by means of a ring which slides freely on the rope. Over how much ground can he feed? What is the length of its outside border? Find these values in square feet and feet, true to six decimal places, when the area of the field is one acre and $a=2b$.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 1.

MECHANICS.

63. Proposed by A. H. BELL, Hillsboro, Ill.

From a horizontal support at a distance of ten feet apart, a beam 5 feet long and 10 pounds weight is suspended by ropes attached to each end. The ropes are 3 and 5 feet respectively, in length. Required the angles made by the ropes and horizontal support. Also the stress upon each rope.

64. Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

Let the isosceles triangle abc, whose plane is vertical, and whose base bc is horizontal, and supported at each end b and c, represent three rods jointed at the points a, b, and c. Let any load L be suspended at the vertex a. It is required to find the value of the angle between the sides of the triangle and the base which shall make the sum of the weights of the rods a minimum, the length of the base bc being fixed.

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 1.

DIOPHANTINE ANALYSIS.

60. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

It is required to find three positive numbers, such that if each be diminished by five times the fifth power of their sum the three remainders will be rational fifth powers.

61. Proposed by SYLVESTER ROBBINS, North Branch, New Jersey.

Investigate that infinite series of prime, integral, rational scalene triangles where the sides of every term are consecutive numbers; then take the necessary factors from the proper KEY, and by an expeditious method, find in their order the areas of ten initial terms.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 1.

AVERAGE AND PROBABILITY.

59. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A circle is rolling along a horizontal straight line. The uniform velocity of the center is v. Find the average velocity of a point of the circumference.

60. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Four points are taken at random within an ellipse. What is the chance that they form a reentrant quadrilateral.

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 1.

MISCELLANEOUS.

58. Proposed by EDMUND FISH, Hillsboro, Ill.

The longest noonday winter shadow of an upright object is found to be seven times as long as the shortest summer shadow of the same object. Required the latitude of the place.

59. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

When a cylindrical china jar, standing upon the ground, receives the sun's rays obliquely, a bright curve is observed to form itself at the bottom of the jar, and it is found that the shape and dimensions of this curve are not affected by the varying elevations of the sun: account for this latter circumstance, and determine the nature of the bright curve. [From *Parkinson's Optics.*]

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 1.

EDITORIALS.

Contributors desiring to have their portraits appear in the next group should send us a good photograph from which the plate is to be made.

We will pay twenty-five cents each for a limited number of Nos. 6 and 11, Vol. II. of the MONTHLY, also fifteen cents for a copy of the October *Cosmopolitan Magazine*.

This number of the MONTHLY is sent to all our old subscribers. Those wishing to discontinue should return this number with their name and address written on the wrapper.

In the February number of the MONTHLY will appear a biography of Bolyai János, by Dr. Halsted. nearly all of which is absolutely new and never before published. Also in addition to Dr. Lovett's valuable article, will appear a New Solution of the Cubic Equation, by Dr. L. E. Dickson.

Dr. Wm. B. Smith, of the Tulane University of Louisiana, informs us that his *Infinitesimal Analysis, Vol. I.* will appear in about two months. Judging from the character of Dr. Smith's previous works, viz., *Introductory to Modern Geometry and Coördinate Geometry*, this work will prove a valuable addition to the literature of the subject.

Prof. W. W. Beman, of the University of Michigan, calls our attention to the fact π was first used to represent the number 3.141592 . . . in Jones' *Synopsis Palmariorum*, London, 1706. This fact seems to have been overlooked by most writers on mathematics as we find the statement that Euler was apparently the first to use the symbol π, *Britannica Encyclopedia*, ninth edition.

The reorganization of the Mathematical Society of Utah took place December 28, 1897, at Provo, electing the following officers : President, W. J. Kerr, President Brigham Young College, Logan, Utah ; Vice President, D. H. Adams, Principal Madison School, Ogden ; Treasurer, G. N. McKay, Principal Lowell School, Salt Lake City ; Secretary, L. M. Gillean, Teacher of Mathematics, Salt Lake High School. The object of the Society is to promote the interest of mathematics in general in the State.

The following letter has been sent us :

To the Editor of THE AMERICAN MATHEMATICAL MONTHLY.

PROPOSED SYLVESTER MEMORIAL.

Sir : May I be permitted to appeal through your columns to all friends and admirers of the late Professor J. J. Sylvester to assist in founding a suitable memoriol in honor to his name and for the encouragement of mathematieal science. A movement was inaugurated on this side of the Atlantic soon after his death and it was resolved by the promoters that a fund should be raised for the purpose of establishing a Sylvester Medal to be awarded at certain intervals for mathematical research to any worker irrespective of nationality. For the purpose of carrying out the scheme a strongly representative International Committee has been formed and I should like to take advantage of this opportunity of expressing the great satisfaction which it has given to the promoters to be enabled to include in this Committee so many great and distinguished names from the Amer-

ican Universities. In every case our invitation to join the Committee has been most cordially responded to and the consent has in many instances been accompanied by expressions of the greatest sympathy and encouragement. The list as it stands practically includes the leading mathematicians of the whole world.

It has been estimated that a capital sum of 5000 dollars will be sufficient for the proposed endowment and of this about one-half has already been subscribed here. In appealing to the American public to enable us to complete the desired sum I am in the first place prompted by the consideration that Sylvester's association with the Johns Hopkins University and the leading part which he took in advancing mathematical science in America renders his claim to estimation on the part of the citizens of your country quite a special one. It is but a modest endowment that we are asking for and I am sure that all those who were personally acquainted with him and who realize the great influence which he exerted in raising the intellectual level of every Institution with which he was associated will be glad of this opportunity of coöperating in the movement.

It is proposed that the fund when complete shall be transferred to the Council of the Royal Society of London, that body having undertaken to accept the trust and to award the medal triennially to mathematicians of all countries. I can hardly venture to trespass upon your courtesy to the extent of asking you to print the complete list of our Committee, but for your own information I beg to send a copy herewith. It will be sufficient to state that it comprises the names of President Gilman of the Johns Hopkins University, of Professor Simon Newcomb of Washington, of Professor Willard Gibbs of Yale, of Professor Peirce of Harvard, and many other well known American men of Science. Subscriptions may be sent to and will be acknowledged by Dr. Cyrus Adler, the Smithsonian Institution, Washington, or by Dr. George Bruce Halsted, President of the Academy of Science, 2407 Guadalupe Street, Austin, Texas.

I am, Sir, Yours obediently,

RAPHAEL MELDOLA,
Professor in the Finsbury Technical College, London, England.
Hon. organizing Secretary to the Sylvester Memorial.

December 1897.

BOOKS AND PERIODICALS.

School Algebra Complete. By Fletcher Durell, Ph. D., and Edward R. Robbins, A. B., Mathematical Masters in the Lawrenceville School. 392 pages. 1898. Harrisburg, Pa.: R. L. Myers & Co.

The effort of the authors has been to simplify principles and make them attractive, by showing as plainly as possible the practical reason for each step or process. The theory of the subject is developed by showing that new symbols are introduced into algebra for the sake of definite advantages in representing numbers, and that the fundamental laws governing their use derive their importance from the economies which they make possible

in dealing with the symbols for numbers. It is here shown that each successive step is taken up for the sake of the new power which it gives as compared with previous processes. While following this line in the development of the subject, no radical departures are made in this book, both the order of topics and the treatment in other respects following closely the best practice as presented in the text-books most used at present.

The selection of examples seems to have been made with excellent judgement and especial care has been exercised in their gradation. The completeness of the book is aided by the addition of valuable short chapters on Permutations and Combinations, Undermined Coefficients, The Binomial Theorem, Continued Fractions, and Logarithms. The treatment in this work is clear and satisfactory, and especially so in the important subjects of factoring, fractions, exponents, and radicals. The authors are practical and successful teachers, and their book contains a very fair presentation of the latest and best methods of treating the subject. J. M. C.

The Equations of Hydrodinamics in a form suitable for application to Problems connected with the Movements of the Earth's Atmosphere. Prepared at the request of Willis L. Moore, Chief of Weather Bureau. By Joseph Collier, Columbia University. Published by authority of the Secretary of Agriculture. Folio Pamphlet, 8 pages.

A New Astronomy for Beginners. By David P. Todd, M. A., Ph. D., Professor of Astronomy and Director of the Observatory, Amherst College. 12mo. Cloth, 480 pages, with colored plates and copious illustrations. Price, $1.30. New York, Cincinnati, and Chicago : American Book Co.

This is by far the most interesting and attractive elementary astronomical work we have seen. Everything in the book is of high scientific and educational value. The illustrations are very artistic and so developed as to give the student an impressive notion of the objects illustrated. We very highly commend this work to all who are looking for a first-class elementary astronomy. B. F. F.

On the Commutator Groups. By Dr. G. A. Miller. Reprinted from the Bulletin of the American Mathematical Society, being a paper read before the Society at its Fourth Summer Meeting, Toronto, Canada, August 17, 1897.

An Algebraic Arithmetic, being an Exposition of the Theory and Practice of Advanced Arithmetic based on the Algebraic Equation. By S. E. Coleman, B. S., William Whitney Fellow at Harvard University, formerly Instructor in Mathematics in the Oakland High School, Oakland, Cal. 8vo. Cloth, 151 pages. Price, 50 cents. New York : The Macmillan Co.

This is another one added to a number of arithmetics recently published which if extensively used in the publish schools will go far towards breaking down the barriers between arithmetic and algebra. This book and others of its kind will be considered by ye pedagogue of ye olden times as an iconoclast. But let it be; from the broken images will rise better results for the mathematics of the future. Of course, it is probable that pupils will always have to commit to memory the multiplication table and to learn other facts about numbers, but when these facts are well fixed in the mind, why continue to manipuiate figures as the symbols of numbers when other and simpler characters may be used to represent a number which is represented by a great many figures? To economize time, to facilitate computations, and to secure better results in teaching elementary mathematics is precisely the purpose of this book. It often makes use of letters to represent number, and introduces the equation from the first. B. F. F.

The Annals of Mathematics. Edited by Wm. H. Echols. Published under the auspices of the University of Virginia. Bi-Monthly, price $2.00 per year in advance.

The October (1897) number of the *Annals of Mathematics* contains the following articles: The Analytical Representation on a Power of Prime Number of Letters with a Discussion of the Linera Group, by Dr. L. E. Dickson; Note on Integral and Integro-Geometrico Series, by Prof. Edward Drake Roe; Note upon a Representation in Space of the Ellipses Drawn by an Ellipsograph, by Prof. E. M. Blake. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

The principal articles of the February number are: The Selection of One's Life Work, by E. Benjamin Andrews; The Great Electric Trust, by Francis Lynde; and Personnel of The Supreme Court, by Nannie-Bille Maury.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

Cuba, Hawaii, and China furnish the principal topics discussed editorially in the *American Monthly Review of Reviews* for February. There are also a few paragraphs of pointed comment on current domestic politics—the factional differences between Ohio Republicans and the swelling tide of Crokerism in the Democratic party. The editor gives his views on Tammany's attitude toward the New York rapid-transit problem and on the reckless expenditure of canal-improvement funds by the Republican bosses of the State.

Some Errata in December Number.

Page 316, line 14, for "$[ac=bd]$" read $[ac+bd]$.
Page 317, lines 20 and 21, in all denominators, for "x^2-y^2" read x^2-yz.
Page 317, line 21, in numerator, for "$[x^2+y^2]$" read $[x^2+yz]$.
Page 317, line 22, for "$[x^2-y^2]$" read $[x^2-yz]$.
Page 318, line 4, for "(1), (5) and (6)" read (1), (10) and (11).
Page 320, Fig. 1 should be reversed in position to correspond to Figs. 2 and 3.
Page 320, line 15, for "$+h\cot[\theta+\alpha]$" read $-h\cot[\theta+\alpha]$.
Page 320, line 18, where "x/z" occurs read $x/2$.
Page 320, line 21, read $[\frac{1}{3}\pi h][x/2]z$.
Page 321, line 14, read $[b-a]^2$ in the first denominator.
Page 322, line 12, for "$\angle DAC$" read $\angle DAB$.
Page 322, line 14, insert) at end of line.
Page 322, in Fig. 3, E and H should be interchanged.
Page 322, line 22, for "$\left(1+\frac{s\sqrt{\beta^2+1}}{a\beta}\right)$" read $\left(1-\frac{s\sqrt{\beta^2+1}}{a\beta}\right)$.
Page 323, line 2, for "$[+\beta]$" read $[k+\beta]$.
Page 323, line 5, insert } before $=$.